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SIMULATION USING THE MONTE CARLO METHOD

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Information Processing Laboratory
Rome Air Development Center
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FOREWORD

The author wishes to express appreciation to Mr. John Kalynycz, Jr., for his assistance in programming the Datatron 205 digital computer. The encouragement and counseling of Dr. M.K. Hu, Professor of Electrical Engineering, Syracuse University, is also much appreciated.

ABSTRACT

An investigation of the effectiveness of the Monte Carlo Method as a simulation technique to be used with a digital computer is carried out in this report.

The problem of communications in a jamming environment is used to illustrate the use of the Monte Carlo Method. Two mathematical models, each representing a different jamming communication configuration, are constructed and solved using the Monte Carlo Method. The first of these models has a known solution which is used to check the Monte Carlo solution.

There are two main conclusions in this report. First, the Monte Carlo Method with the aid of a digital computer can be used to obtain solutions to complex problems quickly and with little effort; second, the use of the Monte Carlo Method does not require the difficult analysis needed for standard or closed techniques.

The report includes a description of the Monte Carlo Method including examples, techniques for generating random numbers, and elements of game theory as applied to jamming communications problems.

PUBLICATION REVIEW

This report has been reviewed and is approved.

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TABLE OF CONTENTS

<i>Contents</i>	<i>Page</i>
I. INTRODUCTION	1
II. THE MONTE CARLO METHOD	1
III. METHODS OF GENERATING PSEUDORANDOM NUMBERS	3
A. Mid-Square Method	3
B. Multiplicative Congruential Method	3
IV. JAMMING AND COMMUNICATION CONFLICT	4
V. A CLOSED SOLUTION OF AN ELEMENTARY JAMMING- COMMUNICATION MODEL	6
A. Transmitter's Characteristics	6
B. Jammer's Characteristics	7
VI. THE MONTE CARLO SOLUTION OF AN ELEMENTARY JAMMING- COMMUNICATION MODEL	9
VII. SOLUTION OF ANOTHER MODEL USING THE MONTE CARLO METHOD	12
A. Transmitter's Characteristics	12
B. Jammer's Characteristics	12
VIII. CONCLUSIONS	17
BIBLIOGRAPHY	18
APPENDIX A - Computer Flow Charts for Model A	19
APPENDIX B - Computer Flow Charts for Model B	21

SIMULATION USING THE MONTE CARLO METHOD

I. INTRODUCTION

The addition of computers as a research tool has made techniques which were once thought of as too lengthy and tedious very powerful for solving complex problems not easily solved by standard or closed methods. One of these techniques is the Monte Carlo Method. The purpose of this investigation is to show the advantage of the Monte Carlo Method over standard techniques in the solution of a complex problem. The problem selected is that of transmitting information in the presence of controlled interference.

II. THE MONTE CARLO METHOD ¹

The term "Monte Carlo" is defined as a technique which consists of simulating some physical model to determine some probabilistic property by the use of random sampling applied to the components of the physical model under investigation. It is not necessary for the physical model to be statistical or probabilistic at the outset, instead these models can be simulated in such a way that a random process can be invented whose statistics provide estimates which describe the behavior of the physical model. Actually, the art of Monte Carlo begins when one abandons a faithful simulation of the process and modifies it so that the characteristic of interest is unchanged, but can be estimated more precisely. The process replaces a determinate problem by a game of chance and the solution for the determinate problem, by the expected score of playing the game. The technique becomes very important when an experiment can be performed much easier than corresponding numerical methods.

The Monte Carlo Method has been used to invert matrices, solve simultaneous equations, solve integral and partial differential equations. This method can best be understood by considering a few examples.

a. First Example: Evaluation of the Integral

$$\int_a^b f(x) dx$$

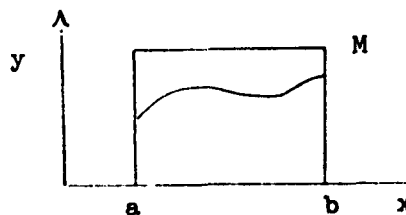


Figure 1.

1. Householder, A.S. Ed., "Monte Carlo Method," Applied Mathematics Series No. 12, Washington, D.C., National Bureau of Standards, 1951, pp. 1-42.

From Figure 1, consider a line $y = M$ such that $M > |f(x)|$ for all x .

$M(b-a) = A$ the area of the rectangle and $I < M(b-a)$

$= A$ where I is the area under the curve $f(x)$

for $a \leq x \leq b$. Now points (x,y) are chosen within the rectangle, such that the choice of points are equally likely. The number of times the point lies in the area I is tabulated. The ratio of success, S (point lying in area I) to total number of tries, N , is the random variable whose expected value is the solution of the integral.

In actual practice, the point in the rectangle can be selected from two rectangular distributions (each point equally likely) in the x and y intervals. With the value of x chosen, the quantity y is compared with $f(x)$ to determine success or failure. According to elementary probabilistic theory, the distribution of S in N trials has an expected value of:

$$\bar{S} = \frac{NI}{A}, \quad (1)$$

with a standard deviation: $\sigma = \sqrt{\frac{NI}{A} \left(1 - \frac{I}{A}\right)}$.

The standard deviation of S/N is:

$$\sigma' = \frac{\sigma}{\sqrt{N}},$$

and the probable error of S/N in percent is then:

$$\text{P.I. (\%)} = 67.45 \sqrt{\frac{1-I/A}{NI/A}}. \quad (2)$$

From equation (2), it is seen that the probable error decreases as N increases and as A decreases. It also indicates that if the quantity of N can be made small, a greater accuracy can be obtained with a smaller number of tries N .

b. Second Example: Random Walk

The problem is to find the distance traveled by a walker after taking n steps from a starting point. The walker's steps are uniform and he may choose with equal probability any one of the four directions: up, down, left, and right. Let us assume an x,y grid with the walker starting at the origin $(0,0)$, and each division representing the length of one step.

Using two separate rectangular distributions, one for selecting steps in the x direction and the other for the y direction, and the following rules, the position of the walker is found:

Rule 1. If a selected number from the distribution governing the x direction is even or zero, let the walker move one step to the right. If this number is odd, let the walker move one step to the left.

Rule 2. If a selected number from the distribution governing the y direction is even or zero, let the walker move one step up. If this number is odd, let the walker move one step down.

Therefore, (X_n, Y_n) represents the position of the walker after the n th step. Now using the distance formula

$$d_n^2 = X_n^2 + Y_n^2,$$

we can determine the distance traveled after the n th step. The operation is repeated several times and the results are averaged to obtain the desired estimate.

III. METHODS OF GENERATING PSEUDORANDOM NUMBERS²

To use the Monte Carlo Method, there must be a way to efficiently generate pseudo-random numbers. A table of numbers could be used but this method would take up too much space in the computing machine and would increase the computing time. It is known that physical processes that assure randomness are those which occur at the atomic or nuclear level. Processes like emission of radio active particles, emission of electrons from a hot filament, motions of gas molecules are random events. The emission of electrons has been used as a source of random numbers in the form of noise generators. This method must contain an analog-digital converter for use in a digital computer, which means using a great part of the computer's memory. To conserve space in the computer, decrease computing time, and make computing more convenient, methods have been devised whereby the computer can generate a random number whenever needed.

A. Mid-square Method

An arbitrary 8-digit number is squared and the middle 8 digits of the product is defined as the new random number. This generated number is again squared with the middle 8 digits of the product becoming another random number. This generation has proven to be inefficient in that numbers like 0000xxxx can be expected well before 10,000 numbers are generated. This will cause vanishing. The method is not restricted to 8-digit numbers.

B. Multiplicative Congruential Method

To produce 8-digit random numbers, the form of this generator is:

$$r_n = K r_{n-1} \pmod{M},$$

with r_0 equal to an arbitrary 8-digit number (starting value)

r_n is the generated number,

K is a multiplier, and

M is $10^8 + 1$.

The value of K equal to 23 produces the maximum period for $M = 10^8 + 1$. This period is 5,882,352.

Example: Let $r_0 = 47594118$,

then $23 \times r_0 = 1094664714$.

2. Householder, op. cit., pp.15-28.

Removing the ninth and tenth digits and subtracting
94664714

- 10

$r_1 = 94664704$ is obtained.

This method is also known as Lehmer's Method.

C. Additive Conguential Method

The form of this generator is:

$$F_{n+2} = F_{n+1} + F_n \pmod{M}$$

with $F_0 = 0$

$$F_1 = 1.$$

Example: Let $F = 8$,

then $F_2 = 1$,

$F_3 = 2$,

$F_4 = 3$,

and so forth.

With $M = 2^{44}$, the method produces a series of 44 binary digits with period 3×2^{43} or 2.5×10^{13} . This method is also known as the Fibonacci Sequence.

It should be noted that these random number generators, with their fixed periods, do not produce sequences that are really random; however, the sequences are sufficiently random depending upon meeting certain statistical requirements. The statistical checks for these pseudorandom sequences will not be discussed. The error involved in using pseudorandom numbers is in the order of $1/\sqrt{N}$ where N is the amount of random numbers generated. Therefore, for results to be correct to 1%, N must equal at least 10,000.

IV. JAMMING AND COMMUNICATION CONFLICT

Communication in the presence of jamming noise contains the elements of a conflict situation and therefore can be set up as a problem in game theory. The communicator will have the choice of selecting one of several strategies or courses of action in a particular time unit with the transmitter also being able to select during this same time unit one of several strategies at its disposal. The solution to the problem is determining the optimum strategy for the communicator to guarantee a minimum loss of transmission regardless what the jammer chooses to do or for the case of the jammer to find the optimum strategy to guarantee a maximum loss of transmission regardless what the communicator chooses to do.

In the two models to be discussed in this study, the measures of benefit will be the ratio of unjammed transmission time to total transmission time. Both transmitter and jammer will use the same transmission techniques, power level and bandwidth. In one of the models, the variable to be chosen at random will be transmission frequency, and in the other model, the random variable will be length of transmissions for the transmitter and jammer. The jammer will be considered as an intelligent player in both cases.

To illustrate the element of conflict, let us look at a zero-sum model³ of a jamming-

3. Williams, J.D., *The Compleat Strategyst*, New York, McGraw-Hill Book Co., Inc., 1954, pp. 29-85.

communication game. This simple game will also bring out some of the points which are presented later in this report. The game is called two person (only two opponents) and zero-sum (that which is gained by one player is lost by the other). Let the transmitter be known as player I and the jammer be player II, and also let each player have two possible alternatives of action: A_1, A_2 for player I and B_1, B_2 for player II. The game will be based on the fact that when player I chooses A_1 , while player II chooses B_1 , the result is favorable to player I to the extent of saving \$100. Likewise, if player I chooses A_2 and is encountered by player II choosing B_2 , the result indicates a \$100 savings for player I. However, whenever A_2 and B_1 or A_1 and B_2 are the strategies chosen, the result is favorable to player II (making player I spend \$100 more).

This game can be described in a game matrix. Each element in the matrix specifies the payoff from player II to player I.

TABLE I
PLAYER II

		PLAYER II	
		B_1	B_2
PLAYER I	A_1	\$100	-\$100
	A_2	-\$100	\$100

The objective of the game is to determine what strategy player I must choose regardless of what his opponent chooses to guarantee a savings of a certain amount V_1 , similarly to determine what strategy player II must choose regardless of the choice of player I to guarantee a loss of a certain amount V_2 to player I. The fundamental principle for game theory states that those optimum strategies exist and that V_1 equals V_2 . This common value V_1 or V_2 is called the value of the game. When the optimum strategies and the value of the game are found, the game is solved.

In this simple game, each player has two possible plays which are weighted with certain probabilities. For this case, the value of the game is interpreted as expected gain of player I averaged over a large number of plays of the game. The specification of a probability distribution over the strategies of one of the players is called a mixed strategy. If one of the players chooses a strategy with probability one, it is called a pure strategy. A saddle point of the game matrix occurs when a pure strategy is the optimum one for both players. The payoff or element of the game matrix at the saddle point is the largest element in the matrix's column and the smallest element in the matrix's row. When both players use their optimum strategies, the game is said to be in equilibrium and any deviation from an optimum strategy by one of the players can be used advantageously by the opponent. Because of this, the intelligent player always plays the game at equilibrium when he is completely unsure of what his opponent will play.

Examining our game matrix, we see that no pure strategy exists for either player. The matrix has no saddle point. By choosing a pure strategy, each player would run the risk of losing \$100, therefore, we will try to find probability distributions that the players can choose at random between the two strategies. Let us assume the following distributions:

player I will choose A_1 with probability p ,
 A_2 with probability $1-p$; and
 player II will choose B_1 with probability q ,
 B_2 with probability $1-q$.

Now if p_o and q_o are optimum values for p and q respectively, the expected value of this game V is:

$$V = p_o q_o (100) + p_o (1-q_o) (-100) + (1-p_o) q_o (-100) + (1-p_o) (1-q_o) (100). \quad (3)$$

In the case of a larger matrix, where players I and II have M and N strategies respectively, then the expression for the value of the game is:

$$V = \sum_{i=1}^M \sum_{j=1}^N p_o^i q_o^j S_{ij}, \quad (4)$$

where

p_o^i is the optimum strategy for player I to strategy number i ,
 q_o^j is the optimum strategy for player II to strategy number j ,

and

S_{ij} is the payoff when II plays number j and I plays number i .

In this example, the optimum probability p_o for player I means that it will guarantee a value at least as great as, if not greater than the amount resulting from using any other probability, likewise, for q_o and player II. Because of this and equations 1 and 2, we find that $p_o = q_o = 1/2$ and the value of the game is equal to zero. That is, that in the long run, neither player will gain by playing the game.

V. A CLOSED SOLUTION OF AN ELEMENTARY JAMMING-COMMUNICATION MODEL

This section describes the closed solution of an elementary jamming-communication model which was derived at Case Institute.⁴ The results obtained from using this method will be used to check the results obtained using the Monte Carlo Method.

Model A

A. Transmitter's Characteristics

1. Transmitter has three choices or strategies:

- a. transmit on frequency A, or
- b. transmit on frequency B, or
- c. monitor the jammer.

These three strategies will be called state A, state B, and state C, respectively.

2. Rules which will govern the transmitter's choice of action:

- a. If the transmitter is in state A or B at $t = t_1$ then the probability of transmitter being in state A or B at $t_1 + 1$ is p and the probability of the transmitter being in state C at $t_1 + 1$ is $(1-p)$.
- b. If the transmitter is in state C at $t = t_1$, then the transmitter's strategy depends on what the jammer's choice was at t_1 .

4. Operations Research Group, Communications in the Presence of an Intelligent Noise Source, Cleveland, Ohio, Case Institute of Technology, 1959, RADC Report No. TR-59-176, pp. 12-17.

- (1) If the jammer was in state A at t_1 , then the transmitter chooses state B with probability 1 at $t = t_1 + 1$.
- (2) If the jammer was in state B at t_1 , then the transmitter chooses state A with probability 1 at $t = t_1 + 1$.
- (3) If the jammer was in state C at t_1 , then the transmitter chooses state A or B with probability 1/2 for each at $t = t_1 + 1$.

B. Jammer's Characteristics

1. The jammer has the same three choices the transmitter has. It can be in state A, B, or C.
2. Rules which will govern the jammer's choice of action:
 - a. If the jammer is in state A or B at $t = t_1$, then the probability that the jammer is in state A or B at $t = t_1 + 1$ is (q) .
 - b. If the jammer is in state C at $t = t_1$, then the jammer's strategy depends on what the transmitter's choice is at $t = t_1$.
 - (1) If the transmitter was in state A at t_1 , then the jammer chooses state A with probability 1 at $t = t_1 + 1$.
 - (2) If the transmitter was in state B at t_1 , then the jammer chooses state B with probability 1 at $t = t_1 + 1$.
 - (3) If the transmitter was in state C at t_1 , then the jammer chooses state C with probability 1 at $t = t_1 + 1$.

Under these circumstances, this model is a nine state system; that is, at any t the system can be in one of nine states (AA, AB, AC, BA, BB, BC, CA, CB, CC), where the first letter indicated the transmitter's state, and the second letter indicates the jammer's state.

It is assumed that the processes under discussion are stationary and therefore, if (p_i) is the probability of a state (i) and $p_i(j)$ the transition probability from state (r) to state (j) , then the following equilibrium condition holds:

$$P_j = \sum_i p_i p_i(j) \quad (5)$$

Not knowing the transitional probabilities of each state to each other and the relationship above, the expected value of each state can be found.

Rewriting (5):

$$\begin{aligned} \text{Probability (given state)} = \\ \sum_{\text{all states}} [\text{prob (each state)} \cdot \text{prob (transition to given state)}] \end{aligned} \quad (6)$$

The transitional probabilities are found using the initial conditions. For an elementary model such as that being examined, this task is relatively simple. With the addition of strategies, the task gets increasingly difficult.

TABLE 2
TRANSITIONAL PROBABILITIES FOR MODEL A

From \ To									
	AA	AB	AC	BA	BB	BC	CA	CB	CC
AA	pq	0	$p(1-q)$	0	0	0	$(1-p)q$	0	$(1-p)(1-q)$
AB	0	pq	$p(1-q)$	0	0	0	0	$(1-p)q$	$(1-p)(1-q)$
AC	p	0	0	0	0	0	$(1-p)$	0	0
BA	0	0	0	pq	0	$p(1-q)$	$(1-p)q$	0	$(1-p)(1-q)$
BB	0	0	0	0	pq	$p(1-q)$	0	$(1-p)q$	$(1-p)(1-q)$
BC	0	0	0	0	p	0	0	$(1-p)$	0
CA	0	0	0	q	0	$(1-q)$	0	0	0
CB	0	q	$(1-q)$	0	0	0	0	0	0
CC	0	0	$1/2$	0	0	$1/2$	0	0	0

$$\begin{aligned}
 \text{Let } X_{11} &= AA & X_{21} &= BA & X_{31} &= CA \\
 X_{12} &= AB & X_{22} &= BB & X_{32} &= CB \\
 X_{13} &= AC & X_{23} &= BC & X_{33} &= CC
 \end{aligned}$$

Now using (6) and the transitional probability table, the following set of equations is obtained:

$$\begin{aligned}
 X_{11} (pq-1) + X_{13} (p) &= 0, \\
 X_{11} p(q-1) + X_{12} p(1-q) - X_{13} + X_{31} (1-q) + X_{33} (1/2) &= 0, \\
 X_{11} q(1-p) + X_{12} q(1-p) + X_{13} (1-p) - X_{31} &= 0, \\
 X_{12} (pq-1) + X_{31} (q) &= 0, \\
 X_{11} 2(1-p)(1-q) + X_{12} (2)(1-p)(1-q) - X_{33} &= 0, \text{ and} \\
 X_{11} + X_{12} + X_{13} + X_{21} + X_{22} + X_{23} + X_{31} + X_{32} + X_{33} &= 1.
 \end{aligned}$$

This set of equations is solved numerically using a digital computer for the given values of (p) and (q) . The payoff in this conflict is the ratio of unjammed transmission over the total transmission time. To obtain this payoff, add up the expected probabilities of the relevant states which are in this game:

$$AB, BA, AC, BC, \text{ or } X_{12}, X_{21}, X_{13}, X_{23}.$$

In summary then, to arrive at a game matrix we:

- . have to arrive at transitional probabilities,
- . construct a system of algebraic equations,
- . solve these equations numerically for various probabilities, and
- . add up the relevant expected values.

Items (1) and (3) become very difficult as the choices for each opponent increases.

There is no automatic means for finding transitional probabilities, and as for the solution of systems of algebraic equations using numerical methods, it is known that as the systems get larger the errors in solution increase.

VI. THE MONTE CARLO SOLUTION OF AN ELEMENTARY JAMMING-COMMUNICATION MODEL

The Monte Carlo Method as used in the solution of the elementary jamming-communication model under investigation, can be described as the use of a determinate probabilistic model approximating an indeterminate physical situation.

Since the physical situation requires that the transmitter and jammer be in one of three states, the model uses a device for generating random numbers which determines the respective state. Also, since the states of the players (jammer and transmitter) are dependent on the previous states, the model is constructed with a starting point (some particular state). Recalling the condition of the physical problem, the process starts as follows: First, the transmitter's previous state is inspected and if it was transmitting on frequency A, a random number is generated and compared with an assigned probability (p) that the transmitter will transmit on frequency A in the next time interval, or probability ($1-p$) that the transmitter will monitor in the next time interval (state C). The same operation applies if the transmitter's previous state was transmitting on frequency B, with the A being replaced by B in the next time interval.

If the transmitter was in state C at the start, then the process required inspecting the jammer's state at the start. If the jammer was set at A or B at the start, then the transmitter will be in state B or A respectively in the next interval. If the jammer was at C at the start, then a random number is generated and compared with $1/2$ to indicate the transmitter being in state A or B in the next interval. The preceding operation will take care of the transmitter's state in the interested interval. The jammer's state for this interval will now be found. The jammer's state at the start is inspected and if it was in state A, a random number is generated and compared with an assigned probability (q) that the jammer will transmit on frequency A in the next time interval or probability ($1-q$) that the jammer will be monitoring (state C). The same operation is performed if the jammer's previous state was B with A being replaced by B in the next time interval. If the jammer was in state C at the start, then the process requires inspecting the transmitter's state at the start. If the transmitter was in state A or B, the jammer will be in state A or B respectively. If the transmitter was in state C, then the jammer will be in state C. The complete time interval is now completed with a transmitter and jammer state. This state is checked for a win or loss to the transmitter. That is, the states AB, AC, BA, BC would score a win for the transmitter; the states AA, BB, CC, CA, and CB would score a loss. The computed interval becomes the previous state and the process is repeated. Each time a win or loss is scored. The problem was solved using 2000, 1000, and 500 intervals. The assigned probabilities (p) and (q) varied from 0 to 1 in step 0.2.

The Datatron 205, a general purpose digital computer, was used to solve the model under comparison. This computer program has 177 steps. Each matrix element was computed in 8, 4, and 2 minutes for 2000, 1000, and 500 intervals respectively.

Table 3 shows the results obtained. Results agree favorably. In each case, the game matrix once reduced, gave the same result. Therefore, using 500 intervals would have been sufficient. This means that a solution to the problem was computed in 72 minutes.

In Table 4, a comparison is made of each of the Monte Carlo solutions to the solution obtained by Case Institute. The deviations tend to show that the error in using the Monte Carlo Method is of the order of $1/\sqrt{N}$, where N is the number of pseudorandom numbers used in the solution.

TABLE 3a
CASE INSTITUTE SOLUTION

p \ q		0	0.2	0.4	0.6	0.8	1.0
0	0.5	0.5	0.4620	0.4460	0.4460	0.4620	0.5000
0.2	0.5	0.5	0.4720	0.4532	0.4418	0.4384	0.4444
0.4	0.5	0.5	0.4926	0.4650	0.4446	0.4176	0.3750
0.6	0.5	0.5	0.4968	0.4878	0.4656	0.4136	0.2856
0.8	0.5	0.5	0.5184	0.5326	0.5304	0.4730	0.1666
1.0	0.5	0.5	0.5556	0.6250	0.7142	0.8334	0 and 1

TABLE 3b
MONTE CARLO 500 INTERVALS

p \ q		0	0.2	0.4	0.6	0.8	1.0
0	0.5	0.5	0.4660	0.4380	0.4500	0.4600	0.5
0.2	0.5	0.5	0.4700	0.4520	0.4400	0.4360	0.4440
0.4	0.5	0.5	0.5040	0.4540	0.4300	0.4200	0.3800
0.6	0.5	0.5	0.4800	0.4980	0.4740	0.4060	0.2840
0.8	0.5	0.5	0.5160	0.5220	0.5300	0.4580	0.1780
1.0	0.5	0.5	0.5560	0.6140	0.7320	0.8220	0 and 1

TABLE 3c
MONTE CARLO 1000 INTERVALS

p \ q		0	0.2	0.4	0.6	0.8	1.0
0	0.5	0.5	0.4620	0.4470	0.4410	0.4670	0.5000
0.2	0.5	0.5	0.4780	0.4600	0.4390	0.4360	0.4430
0.4	0.5	0.5	0.4810	0.4610	0.4420	0.4220	0.3710
0.6	0.5	0.5	0.5070	0.4910	0.4650	0.4080	0.2820
0.8	0.5	0.5	0.5160	0.5360	0.5270	0.4750	0.1710
1.0	0.5	0.5	0.5520	0.6370	0.7110	0.8280	0 and 1

TABLE 3d
MONTE CARLO 2000 INTERVALS

p \ q		0	0.2	0.4	0.6	0.8	1.0
0	0.5	0.5	0.4630	0.4445	0.4480	0.4670	0.5000
0.2	0.5	0.5	0.4760	0.4590	0.4460	0.4385	0.4470
0.4	0.5	0.5	0.4820	0.4650	0.4455	0.4225	0.3740
0.6	0.5	0.5	0.5020	0.4905	0.4650	0.4350	0.2820
0.8	0.5	0.5	0.5170	0.5385	0.5300	0.4705	0.1655
1.0	0.5	0.5	0.5535	0.6220	0.7175	0.8405	0 and 1

TABLE 4a
COMPARISON OF THE CASE SOLUTION WITH
THE MONTE CARLO 500 INTERVAL SOLUTION

p \ q						
	0	0.2	0.4	0.6	0.8	1.0
0	0	-0.0040	0.0080	-0.0040	0.0020	0
0.2	0	0.0020	0.0012	0.0018	0.0024	0
0.4	0	-0.0114	0.1100	0.0140	-0.0024	-0.0050
0.6	0	0.0168	-0.0102	-0.0084	0.0076	0.0014
0.8	0	0.0024	0.0106	0.0004	0.0150	-0.0114
1.0	0	-0.0004	0.0110	0.0178	0.0114	0

TABLE 4b
COMPARISON OF THE CASE SOLUTION WITH
THE MONTE CARLO 2000 INTERVAL SOLUTION

p \ q						
	0	0.2	0.4	0.6	0.8	1.0
0	0	-0.0010	0.0015	-0.0020	-0.0050	0
0.2	0	-0.0040	-0.0008	-0.0042	-0.0001	-0.0026
0.4	0	0.0106	0	-0.0009	-0.0049	0.0010
0.6	0	-0.0052	-0.0027	0.0006	-0.0214	0.0036
0.8	0	0.0014	-0.0059	0.0004	0.0025	0.0011
1.0	0	0.0021	-0.0030	-0.0032	-0.0071	0

TABLE 4c
COMPARISON OF THE CASE SOLUTION WITH
THE MONTE CARLO 1000 INTERVAL SOLUTION

p \ q						
	0	0.2	0.4	0.6	0.8	1.0
0	0	0	-0.0010	0.0050	-0.0050	0
0.2	0	-0.0060	-0.0068	0.0028	0.0024	0.0014
0.4	0	0.0116	0.0040	0.0026	-0.0044	0.0040
0.6	0	-0.0102	-0.0032	0.0006	0.0056	0.0036
0.8	0	0.0024	-0.0034	0.0034	-0.0020	-0.0044
1.0	0	0.0036	-0.0120	0.0032	0.0054	-0.020

VII. SOLUTION OF ANOTHER MODEL USING THE MONTE CARLO METHOD

The preceding process was applied to another jamming-communication model in which the length of transmission is the random variable. The purpose of investigating this problem is to determine the best distribution for the random variable. This condition would be one in which the jammer ensures at best some portion of lost time and the transmitter ensures that nothing the jammer could do would increase the lost time.

Since the physical situation requires that any sequence of jammer time or transmitter time shall have the different message lengths distributed randomly, the model uses a device for generating random numbers to determine which message length shall be used.

Model B

The configuration in this model consists of a transmitter sending out messages to a receiver with preprogramed frequency and message length selection, while an intelligent jammer attempts to prevent the receiver from receiving the messages.

A. Transmitter's Characteristics

1. The transmitter can select a message length of 1 unit or 5 units of time. The selection of either of the two message lengths is determined by an assigned probability x of choosing 5 units of message and probability $1-x$ of choosing 1 unit of message.
2. The transmitter has two frequencies on which to transmit f_1 and f_2 each of which are chosen with equal probability. The choice of changing frequency will be made after each of the transmitter's transmission.
3. Switching time needed for transmitter to change frequency is 1 unit of time.

B. Jammer's Characteristics

1. The jammer can select a message length of 1 unit or 5 units of time. The selection of either of the two message lengths is determined by an assigned probability y of choosing 5 units of messages, and probability $1-y$ of choosing 1 unit of message.
2. The jammer has the same two frequencies as the transmitter on which to transmit, the choice is dependent on the transmitter's choice.
3. Searching time for the jammer to monitor the transmitter's frequency is 1 unit. The jammer will search after each of its transmissions.
4. The jammer is considered intelligent, if during the search unit the jammer discovers it is off frequency with the transmitter, it will change frequency in the next time unit; if during the search the jammer finds the transmitter in a switch state, the jammer will search again in the next time unit.

For each unit of time, the transmitter can be in one of six states:

State	Description
S	Just completed a frequency switch
M_1	Just completed 1 unit of message
M_2	Just completed 2 units of message
M_3	Just completed 3 units of message
M_4	Just completed 4 units of message
M_5	Just completed 5 units of message

and the jammer can be in one of seven states:

State	Description
H	Just completed a search
S	Just completed a frequency switch
J_1	Just completed 1 unit of message
J_2	Just completed 2 units of message
J_3	Just completed 3 units of message
J_4	Just completed 4 units of message
J_5	Just completed 5 units of message

In any particular time unit, the transmitter and jammer can be in one of the 42 different combinations.

The problem is now simulated as follows: Initially, the jammer and transmitter start with the same frequency. The probabilities x and y for the transmitter and jammer to broadcast 5 time units respectively are initially set to zero. A random number is generated and compared to 0.5. If the number is less or equal to 0.5, the transmitter will change frequency after its message length; if the number is greater than 0.5, there will be no frequency switch after the transmitter's message. Another random number is generated and compared with (x) to determine whether the transmitter will broadcast 1 or 5 time units of message. The preceding operation can lead us to four separate message sequences for the transmitter:

$.ma$	1 unit of message + 1 unit for frequency switch = 2 time units,
$.m$	1 unit of message = 1 time unit,
$.mmmmms$	5 units of message + 1 unit for frequency switch = 6 time units,
$.mmmmm$	5 units of message = 5 time units.

Now looking at the jammer, we find the procedure dependent on the transmitter's strategy. The jammer is initially set on the same frequency as the transmitter, and a random number is generated and compared with y to determine whether 1 or 5 time units of message are sent out. A search is made after each jamming transmission. The jammer's choice will be:

$.mh$	1 unit of message + 1 unit for search = 2 time units,
or $.mmmmh$	5 units of message + 1 unit for search = 6 time units.

Now if the jammer is searching while the transmitter is switching frequency, the jammer must follow with another search. To better understand the condition, let us look at a typical sequence. Let the numerator of the following ratio be the transmitter's sequence and the denominator represent the jammer's sequence:

Now if the following condition existed:

$$\frac{ms}{mh},$$

the jammer must next follow with another search and the sequence would be:

$$\frac{ms}{mhh}$$

The jammer is considered intelligent in that during the search time, it can determine whether it is on the same frequency as the transmitter. If it is not on frequency, it will switch in the next time unit:

Example (transmitter and jammer not on frequency)

$$\frac{mmm}{mhs}.$$

To make the model easier to program for a digital computer, the jammer's search is conducted after its message transmission and the information on whether it is on or off the transmitter's frequency is held in memory to be used for the next simulated transmission or sequence.

Each time unit is checked to see if the transmitter gets through the jammer. The acceptable states are:

$$\frac{m_1}{m_2}, \frac{m_2}{m_1}, \frac{m_1}{h}, \frac{m_2}{h}, \frac{m_1}{s}, \frac{m_2}{s},$$

where

m_1 is the message unit at f_1 , and

m_2 is the message unit at f_2 .

The states which indicate lost transmitter time are:

$$\frac{m_1}{m_1}, \frac{m_2}{m_2}, \frac{s}{m_1}, \frac{s}{m_2}, \frac{s}{h}, \frac{s}{s}.$$

The result of unjammed time to total transmission time is recorded. The problem was solved using 2000 separate message sequences and for the following probabilities of x and y .

$x = 0.$ to 1.0 with $\Delta x = 0.1$, and

$y = 0.$ to 1.0 with $\Delta y = 0.1$.

The results are shown in matrix form in Table 5.

To further simplify the explanation of this process, let us follow a few sequences that might occur. The model starts with the jammer initially on the same frequency as the transmitter and at t_0 the jammer is searching.

1. Initially: $\frac{\text{transmitter}}{\text{jammer}} \quad \frac{t_0 \ t_1 \ t_2 \ t_3 \ t_4 \ t_5 \ t_6}{h}$

2. A random number is generated and compared to 0.5 to determine whether the transmitter will change frequency at the end of its message length. Another random number is generated and compared with x to determine whether 1 or 5 units of message are sent out by the transmitter. If a message of 1 unit with no frequency change is

selected, the result would be:

$$\frac{m}{h}.$$

3. This state is checked for success or failure to the transmitter and recorded. In this example, there is a message for the transmitter.

4. Now a random number is generated and compared to γ to determine the jammer's choice of 1 or 5 units of message. If the choice was 5 units, the result would be:

$$\frac{m}{h \ m \ m \ m \ m \ h}.$$

5. Repeating step 2 and assuming the transmitter's selection is 5 units with a change of frequency, then:

$$\frac{m \ m \ m \ m \ m \ s}{h \ m \ m \ m \ m \ h}.$$

6. States $t_1, t_2, t_3, t_4, t_5, t_6$ are checked for success or failure and recorded.

7. Since the jammer is searching while the transmitter is changing frequency in t_6 , the jammer will automatically be in the search state at t_7 .

$$t_0 \ t_1 \ t_2 \ t_3 \ t_4 \ t_5 \ t_6 \ t_7 \ t_8 \ t_9 \ t_{10} \ t_{11} \ t_{12} \ t_{13} \ t_{14} \ t_{15} \ t_{16}$$

$$\frac{m \ m \ m \ m \ m \ s}{h \ m \ m \ m \ m \ h \ h}.$$

8. Repeating step 2 and assuming the transmitter will change frequency and send out 5 units, then:

$$\frac{m \ m \ m \ m \ m \ s \ m \ m \ m \ m \ s}{h \ m \ m \ m \ m \ h \ h \ s}.$$

9. State t_7 is checked for success or failure and then step 4 is repeated. Assuming the jammer's selection is 1 message unit, then:

$$\frac{m \ m \ m \ m \ m \ s \ m \ m \ m \ m \ s}{h \ m \ m \ m \ m \ h \ h \ s \ h}.$$

10. States t_8, t_9, t_{10} are checked for success or failure and since the states t_{11}, t_{12} are not complete, step 4 is repeated again. Assume the jammer's selection is 5 units, then:

$$\frac{m \ m \ m \ m \ s}{h \ s \ h \ m \ m \ m \ m \ h}.$$

11. Step 2 is repeated. It is assumed the transmitter will send out 1 unit with no frequency change, then:

$$\frac{s \ m}{m \ m \ m \ h}.$$

12. State t_{13} is checked for success or failure and then step 2 is repeated. Assume the transmitter sends out 5 units with no frequency change, then:

$$\begin{array}{ccccccc}
 t_{12} & t_{13} & t_{14} & t_{15} & t_{16} & t_{17} & t_{18} \\
 s & m & m & m & m & m & m \\
 \hline
 & m & m & m & m & h &
 \end{array}$$

13. States t_{14} , t_{15} , t_{16} are checked for success or failure and so on.

Table 5 is analyzed for reduction to a usable size. First, a check is made for a saddle point. There is none. Checking the rows, it is observed that the rows for $x = 0.2$ to 1.0 can be neglected since the majority of elements in these rows are less than the values for rows $x = 0, 0.1$. Checking the columns of these two rows, it is found that the elements in columns $y = 0.9, 1.0$ are less than the other elements. This operation reduces the matrix to a 2×2 system:

x \ y		
	0.9	1.0
0	0.4436	0.4518
0.1	0.4464	0.4333

From this matrix, it can be concluded that the best strategy for the transmitter is to transmit message lengths of 1 unit duration with probability one, and the best strategy for the jammer is to broadcast message lengths of 5 units duration with probability one.

A similar computer run was contemplated for a non-preprogramed transmission, where the receiver is not necessarily always on the same frequency as the transmitter. This would entail that the transmitter must identify itself before transmission, resulting in more lost time than the preprogramed case. The effort involved in obtaining the solution to this case was considered too great and would not add to the purpose of this report.

TABLE 5

x \ y	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	.554	.521	.516	.498	.484	.484	.475	.479	.458	.443	.451
0.1	.552	.530	.506	.500	.464	.490	.473	.443	.446	.446	.433
0.2	.546	.510	.493	.490	.477	.441	.456	.446	.424	.413	.433
0.3	.542	.518	.500	.463	.480	.454	.464	.420	.451	.411	.427
0.4	.537	.513	.497	.466	.461	.461	.451	.424	.431	.411	.407
0.5	.532	.500	.485	.469	.461	.445	.471	.425	.426	.399	.412
0.6	.531	.507	.491	.467	.442	.420	.447	.426	.405	.416	.406
0.7	.524	.495	.463	.469	.432	.448	.432	.423	.431	.431	.406
0.8	.520	.481	.495	.447	.461	.457	.444	.418	.425	.419	.429
0.9	.521	.479	.473	.462	.454	.431	.438	.440	.421	.437	.425
1.0	.515	.475	.458	.453	.446	.452	.443	.444	.437	.435	.417

VIII. CONCLUSIONS

The work presented in this report does not consider existing jamming-communication situations. The purpose was to examine workable models in order to investigate a technique which can be used advantageously for studying the more complicated problems of physical systems.

The results obtained show that with the use of the Monte Carlo Method, the solution of an elementary model is obtained more quickly while still maintaining the accuracy of the standard techniques. The study also shows that as the problem becomes more complex, the Monte Carlo Method becomes more advantageous for obtaining a solution. The accuracy of the method is dependent on the accuracy obtained in generating pseudo-random numbers which is stated as of the order $1/\sqrt{N}$ where N is the number of pseudo-random numbers used to obtain a solution.

Another important advantage of using this technique is that difficult analysis based upon probability theory is not required. The report discussed the need of obtaining transitional probabilities to arrive at a solution using the standard techniques. It can be seen that as the number of possible strategies increases, the task of deriving transitional probabilities gets more difficult.

For those jamming-communication situations which are very complex and are almost impossible to solve with standard techniques, the Monte Carlo Method can be used to obtain quick estimates of the solution, which in turn can enable one to modify his original model to a simpler one, and then use the standard techniques to obtain a refinement of the solution.

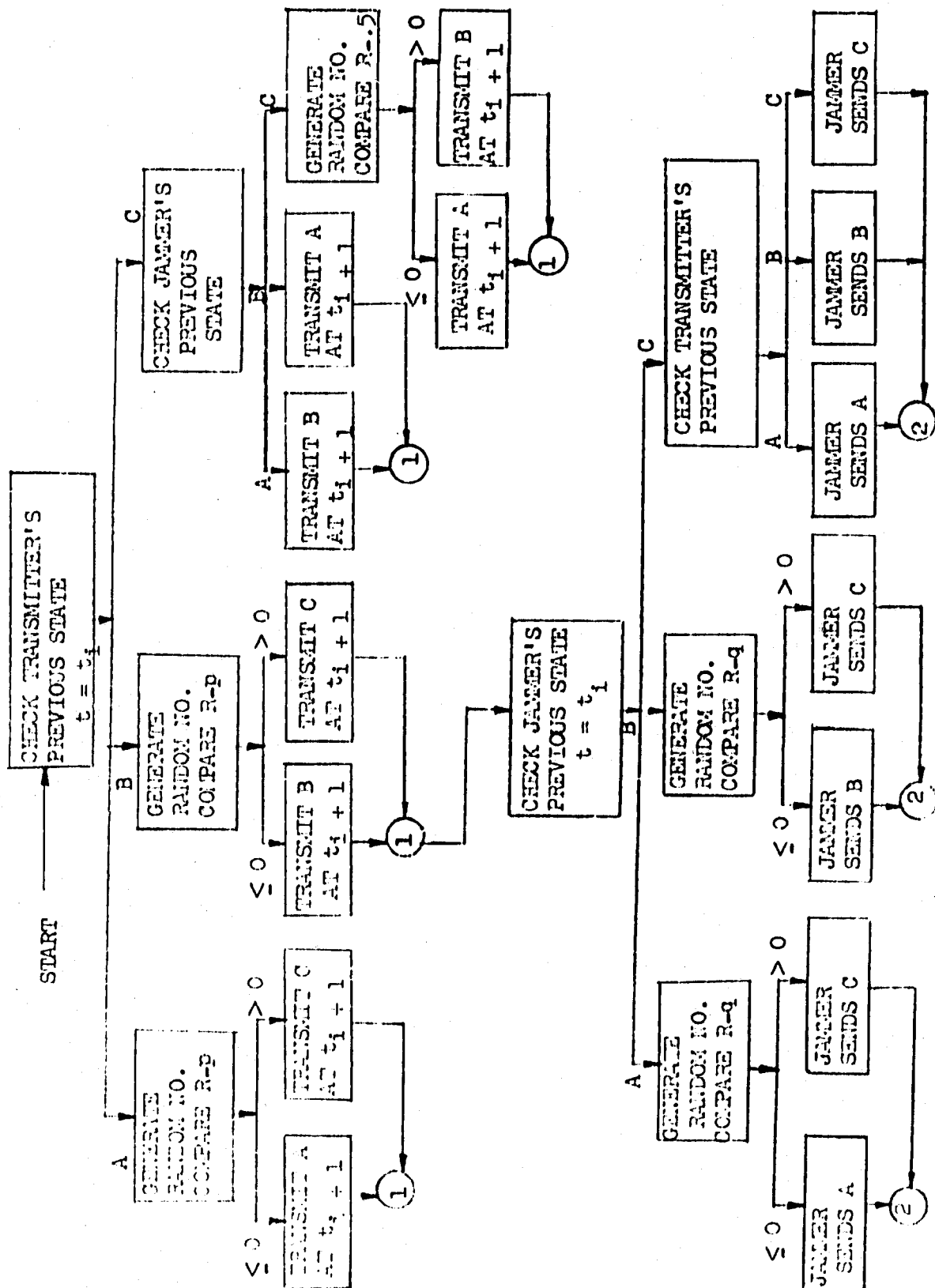
In general, the significance of the Monte Carlo Method is due to the relatively slow increase in the work required for a solution with increasing numbers of equations and unknowns.

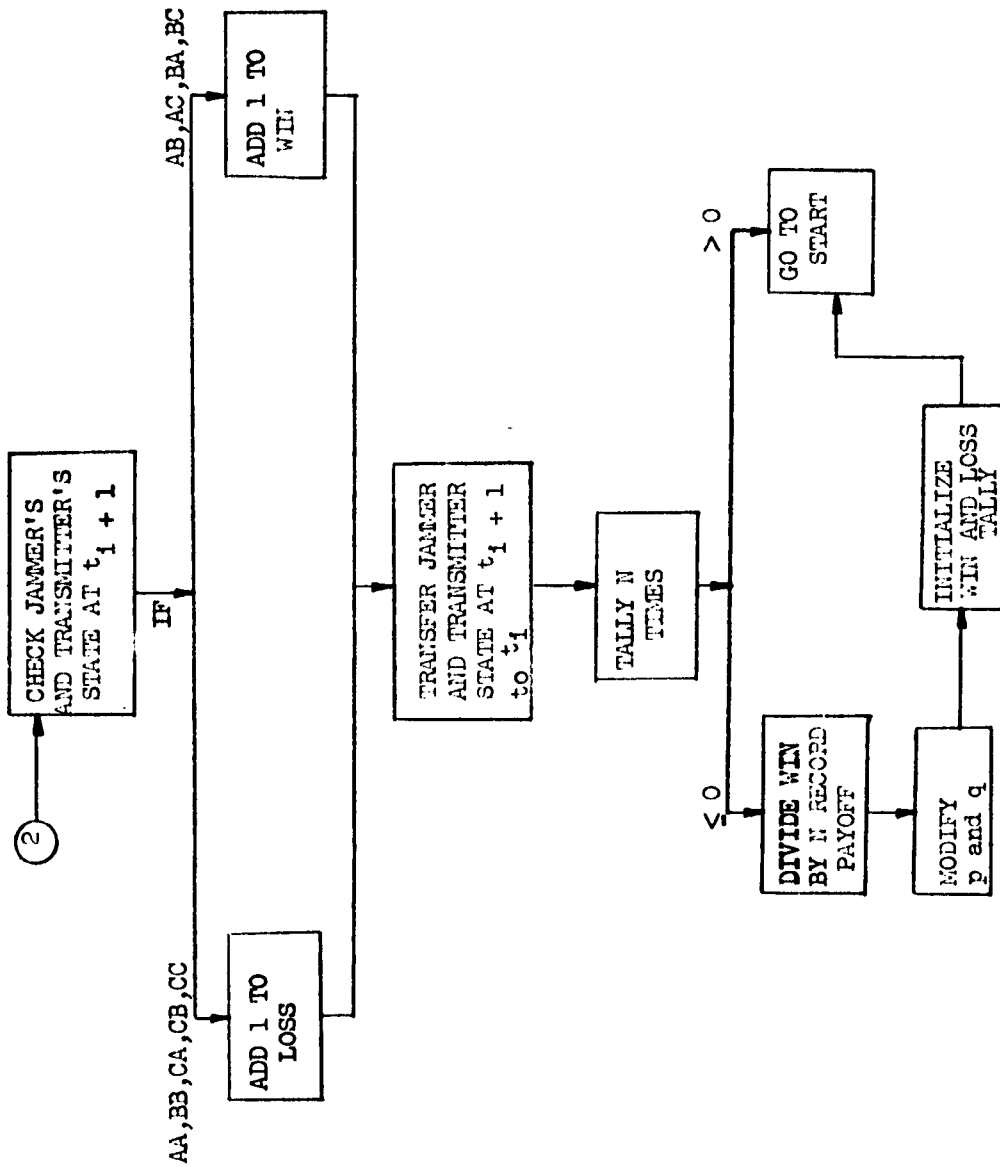
The Monte Carlo Method as discussed in this report is not limited to the solution of jamming-communication problems. It can be used successfully to solve other physical situations or problems which are too complicated to be treated by deterministic methods.

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APPENDIX A
COMPUTER FLOW CHARTS FOR MODEL A





APPENDIX B
COMPUTER FLOW CHARTS FOR MODEL B.

